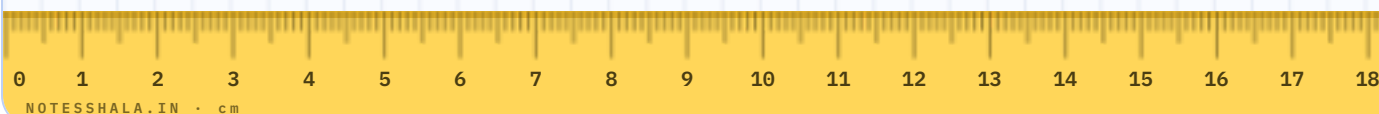


CHAPTER

06

System of Particles and Rotational Motion

Rigid Body and Centre of Mass • Torque and Angular Momentum •
Equilibrium and Centre of Gravity • Moment of Inertia and Rotational
Dynamics



What's inside

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READING KEY

DEFINITION

NOTE

REMEMBER

DERIVATION

STRATEGY

Formulas sit on graph paper • coloured words are the key terms

Till now, we treated every object as a **point particle**. A real body has a size. It is made of a very large number of particles. So it is called an **extended body**. In this chapter, we study the motion of a **system of particles** and of a **rigid body**. The main ideas are **centre of mass**, **torque**, **angular momentum**, **moment of inertia** and **rolling motion**. Rolling is a mix of translation and rotation.

1.1 Rigid Body and Types of Motion

DEFINITION

A **rigid body** is a body in which the **distance between every pair of particles stays fixed**, however large the force on it.

No real body is perfectly rigid. But real bodies change their shape only a little. So most real bodies can be taken as **rigid bodies**.

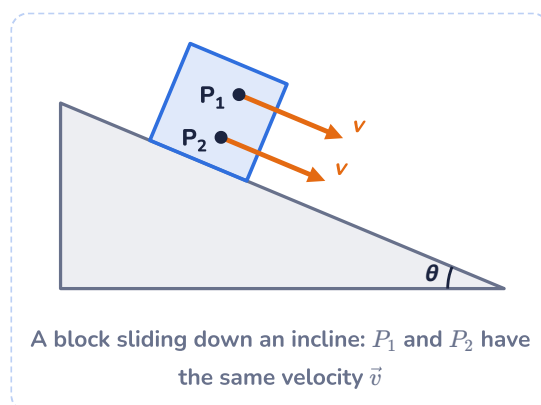
There are **two basic kinds of motion** of a rigid body: **pure translation** and **pure rotation**. A body can also do both together.

• PURE TRANSLATIONAL MOTION

During **pure translation**, every particle of the body moves with the **same velocity** at any instant. So in a given time, all the particles move through the **same displacement**.

A block **sliding** (not rolling) down an inclined plane

Translation



• ROTATION ABOUT A FIXED AXIS

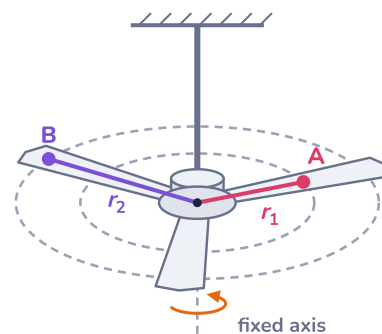
DEFINITION

Axis of Rotation

In **pure rotation**, every particle of the body moves in a **circle**. All these circles have their centres on one fixed straight line. This line is called the **axis of rotation**.

Look at a rotating **ceiling fan**. A particle A on a blade moves in a circle of radius r_1 . A particle B moves in a bigger circle of radius r_2 .

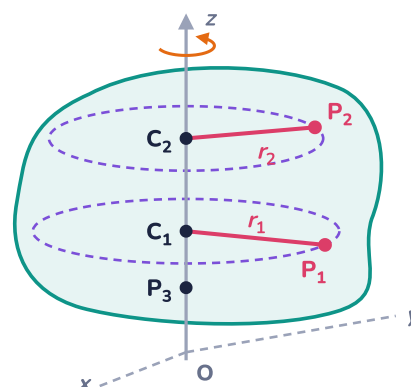
The centres of both circles lie on the **same fixed axis**. This axis is the vertical line through the rod of the fan.



A rotating ceiling fan: A and B move in circles about the same axis

Take a particle P_1 at a distance r_1 from the axis. It moves in a circle of radius r_1 . The centre C_1 of this circle lies **on the axis**.

This circle lies in a plane **perpendicular to the axis**. In the same way, a particle P_2 moves in a circle of radius r_2 with centre C_2 .



A rigid body rotating about the z -axis

i Different Planes

The circles of P_1 and P_2 need not be in the **same plane**. Still, each of these planes is **at right angles to the fixed axis**.

ii On the Axis

A particle on the axis, like P_3 , has $r = 0$. So it **stays at rest** while the body rotates.

NOTE

- Each particle moves in a circle whose plane is **perpendicular to the fixed axis**. The **centre of this circle is on the axis**.
- A particle that lies on the axis of rotation **does not move**.

e.g. **Potter's wheel, giant wheel, merry-go-round, ceiling fan**

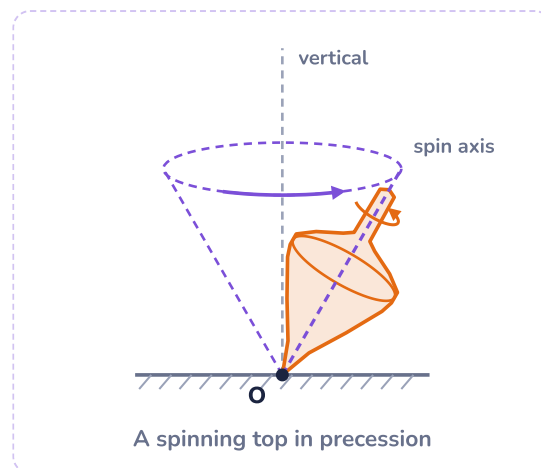
Rotation about a fixed axis

• PRECESSION

DEFINITION

Precession is the motion of an axis of rotation that turns around another line. **One point of the axis** stays fixed. So the axis sweeps out a **cone**.

- ◆ A **spinning top** shows precession. Its pointed tip touches the ground and stays **fixed**. At every instant, the axis of rotation passes through this point.
- ◆ An **oscillating table fan** also shows precession.



REMEMBER

Convention

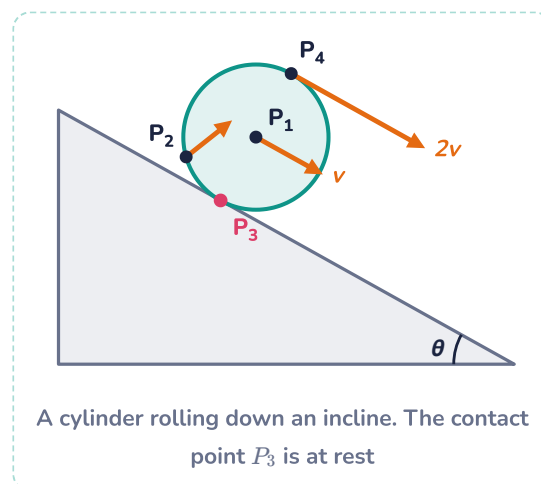
In this chapter, we mostly study rotation about a **fixed axis**. So take every rotation to be about a fixed axis, unless the question says something else.

1.2 Combination of Translation and Rotation

A rigid body can also **roll**. Rolling is a **combination of rotation and translation**. Take a **cylinder rolling down an inclined plane**. It turns about its own axis. At the same time, this axis moves down the slope.

As the cylinder rolls down, its points P_1 , P_2 , P_3 and P_4 have **different velocities** at the same instant.

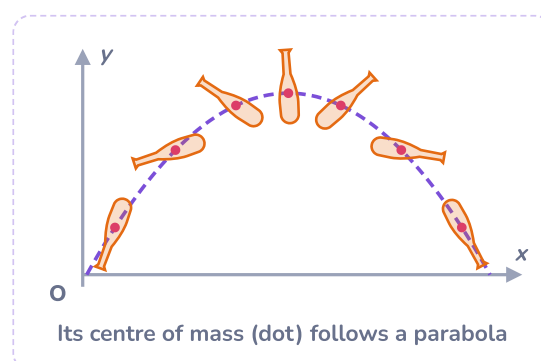
If the cylinder **rolls without slipping**, the point of contact P_3 has **zero velocity** at every instant.



So in rolling, the body turns about an **axis of rotation**, and this axis itself moves in **translation**. So any motion of a rigid body is of one of two kinds. It is **pure translation**, or it is **translation and rotation together**.

A **baseball bat** thrown into the air is a good example. Its motion is **not pure translation**. It is **not pure rotation** either. It is a **combination of both**.

The bat keeps turning. But one special point of the bat, its **centre of mass**, follows a smooth parabola.



NOTE

A rigid body that is **not pivoted or fixed** in any way

Pure translation, or translation + rotation

A rigid body that is **pivoted or fixed** in some way

Pure rotation

1.3 Centre of Mass of Two Particles

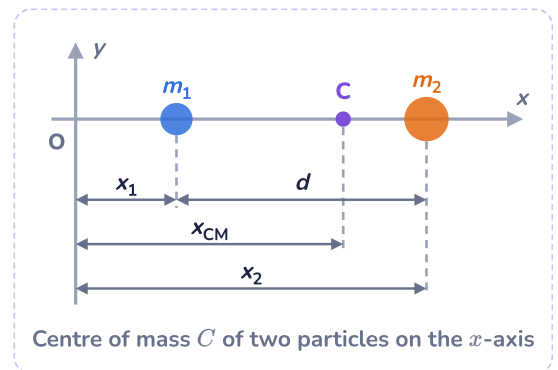
DEFINITION

The **centre of mass** (CM) of a system is the point where the **whole mass of the system** can be taken to be concentrated.

The system moves as if all its mass were placed at this point. All the external forces also seem to act at this point.

Two particles of masses m_1 and m_2 lie on the x -axis. Their distances from the origin O are x_1 and x_2 .

Their centre of mass is the point C . Its distance from O is x_{CM} .



CENTRE OF MASS (TWO PARTICLES)

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m_1 x_1 + m_2 x_2}{M}$$

where, $M = m_1 + m_2$ = total mass of the system

i Position Vector Form

Multiply the position vector of the CM by the total mass. The result equals the sum of each mass times its own position vector.

$$M \vec{r}_{CM} = m_1 \vec{r}_1 + m_2 \vec{r}_2$$

ii Equal Masses

If $m_1 = m_2 = m$, then:

$$x_{CM} = \frac{x_1 + x_2}{2}$$

So the CM is at the **mid-point** of the line joining them.

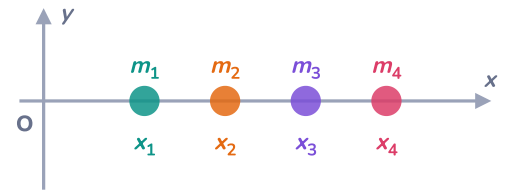
x_{CM} is the **mass-weighted average** of x_1 and x_2 . The CM lies on the line joining the two particles

On the line

1.4 Centre of Mass of n Particles

Take n particles of masses $m_1, m_2, \dots, m_n$ on the x -axis. Their positions are $x_1, x_2, \dots, x_n$.

The x -coordinate of their CM is the **mass-weighted average** of all these positions.



A system of particles on the x -axis

CENTRE OF MASS (N PARTICLES)

$$x_{CM} = \frac{m_1x_1 + m_2x_2 + \dots + m_nx_n}{m_1 + m_2 + \dots + m_n} = \frac{\sum m_i x_i}{\sum m_i}$$

• CENTRE OF MASS OF THREE PARTICLES

Three particles of masses m_1, m_2 and m_3 lie at the points (x_1, y_1) , (x_2, y_2) and (x_3, y_3) . Their CM is at the point (X, Y) :

i Any Masses

$$X = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$Y = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3}$$

ii Equal Masses ($m_1 = m_2 = m_3 = m$)

$$X = \frac{x_1 + x_2 + x_3}{3}$$

$$Y = \frac{y_1 + y_2 + y_3}{3}$$

• IN THREE DIMENSIONS

When the particles are spread in **three-dimensional space**, the CM has three coordinates:

X-COORDINATE

$$x_{CM} = \frac{1}{M} \sum m_i x_i$$

Y-COORDINATE

$$y_{CM} = \frac{1}{M} \sum m_i y_i$$

Z-COORDINATE

$$z_{CM} = \frac{1}{M} \sum m_i z_i$$

POSITION VECTOR OF CENTRE OF MASS

$$\vec{r}_{CM} = \frac{1}{M} \sum_{i=1}^n m_i \vec{r}_i \quad \left[M = \sum_{i=1}^n m_i \right]$$

If all the n masses are **equal**, the CM is just the **average of their positions**. For example, $X = \frac{x_1 + x_2 + x_3}{3}$.

1.5 Centre of Mass of a Continuous Body

In a real body, matter is spread out **continuously**. It is not a set of separate particles. So we divide it into tiny pieces of mass. Then the sum over the particles becomes an **integral**.

• POSITION VECTORS AND THE CM

i Vector Form

Let $\vec{r}_i = x_i\hat{i} + y_i\hat{j} + z_i\hat{k}$ be the position vector of the i^{th} particle. Let $\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$ be the position vector of the CM. Then:

$$\vec{R} = \frac{\sum m_i \vec{r}_i}{M}$$

ii Origin at the CM

Choose the origin **at the centre of mass**. Then $\vec{R}_{CM} = 0$. So:

$$\sum_{i=1}^n m_i \vec{r}_i = 0$$

CONCEPT

CM of a Continuous Body

In a continuous body, each point mass becomes a tiny **mass element** dm . Each sum becomes an **integral**. So the coordinates of the CM are:

CENTRE OF MASS (INTEGRAL FORM)

$$x_{CM} = \frac{1}{M} \int x dm, \quad y_{CM} = \frac{1}{M} \int y dm, \quad z_{CM} = \frac{1}{M} \int z dm$$

where, M = total mass of the body. If the origin is at the CM, then $\int x dm = \int y dm = \int z dm = 0$.

• CM WHEN NO EXTERNAL FORCE ACTS

Suppose **no external force** acts on a system. If its CM is at rest at first, the CM **stays at the same place**.

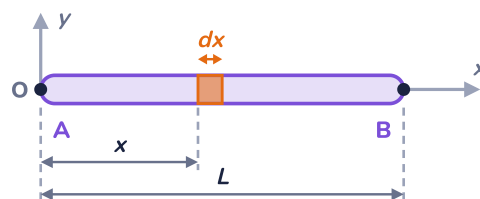
If the particles shift by $\Delta x_1, \Delta x_2$ and so on, the CM shifts by:

$$\Delta x_{CM} = \frac{\sum m_i \Delta x_i}{\sum m_i}$$

1.6 Centre of Mass of a Uniform Thin Rod

A **thin uniform rod** AB has mass M and length L . It lies along the x -axis, with end A at the origin.

Take a small element of length dx at a distance x from A .



A small element dx at distance x from A

DERIVATION

- 1 As the rod is uniform, its mass per unit length is the same everywhere. This is its **linear mass density**, $\lambda = \frac{M}{L}$. So the mass of the small element is:

$$dm = \frac{M}{L} dx$$

- 2 The x -coordinate of the CM is:

$$x_{CM} = \frac{1}{M} \int_0^L x dm = \frac{1}{M} \int_0^L x \left(\frac{M}{L} \right) dx = \frac{1}{L} \left[\frac{x^2}{2} \right]_0^L = \frac{L}{2}$$

- 3 The rod lies on the x -axis. So $y = 0$ for every element:

$$y_{CM} = \frac{1}{M} \int y dm = 0 \quad [\because y = 0]$$

CENTRE OF MASS OF A UNIFORM ROD

$$x_{CM} = \frac{L}{2}, \quad y_{CM} = 0$$

So the CM is at $\left(\frac{L}{2}, 0 \right)$, exactly midway between A and B . For a **uniform body**, the CM lies at its **geometric centre**.

1.7 Motion of Centre of Mass

How the CM moves depends on how each particle moves. Start from the equation that defines the CM. Then **differentiate it with respect to time**. This gives the velocity and the acceleration of the CM.

$$M \vec{r}_{CM} = m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 + \cdots + m_n \vec{r}_n$$

DERIVATION

- 1 Differentiate once with respect to time. The masses are constant. Only the position vectors change:

$$M \frac{d}{dt} \vec{r}_{CM} = m_1 \frac{d\vec{r}_1}{dt} + m_2 \frac{d\vec{r}_2}{dt} + \cdots + m_n \frac{d\vec{r}_n}{dt}$$

- 2 Rate of change of position = velocity, that is $\vec{v} = \frac{d\vec{r}}{dt}$. So:

$$M \vec{v}_{CM} = m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3 + \cdots + m_n \vec{v}_n$$

3 Differentiate again. Rate of change of velocity = acceleration, that is $\vec{a} = \frac{d\vec{v}}{dt}$. So:

$$M \vec{a}_{CM} = m_1 \vec{a}_1 + m_2 \vec{a}_2 + \cdots + m_n \vec{a}_n$$

VELOCITY OF CENTRE OF MASS

$$\vec{v}_{CM} = \frac{\sum_{i=1}^n m_i \vec{v}_i}{M}$$

ACCELERATION OF CENTRE OF MASS

$$\vec{a}_{CM} = \frac{\sum_{i=1}^n m_i \vec{a}_i}{M}$$

NET EXTERNAL FORCE ON A SYSTEM

$$M \vec{a}_{CM} = \vec{F}_{ext}$$

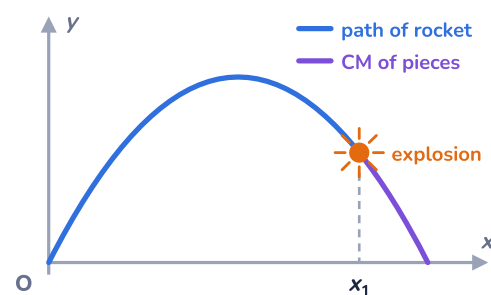
The CM moves as if the **whole mass** of the system were placed at it, and **all the external forces** acted on it there.

INTERNAL FORCES DO NOT AFFECT THE CM

Every **internal force** has an equal and opposite partner. These are **action–reaction pairs**. So the internal forces add up to zero, and only the **external forces** remain in the sum.

Suppose a **rocket explodes** in mid-air. The forces of the explosion are internal. So they add **nothing** to the acceleration of the CM.

The CM of the pieces keeps moving on the **same parabola**. This is the path of the rocket without the explosion.



After the explosion, the CM of the pieces stays on the same parabola

NOTE

The **translational motion** of an extended body is easy to study through its CM. Imagine that the **total mass sits at the CM**, and that every external force acts on that point.

1.8 Linear Momentum of a System of Particles

DERIVATION

To get the **total linear momentum** of a system, add the momenta of all its particles as **vectors**. One particle has momentum $m\vec{v}$.

1 Add the momenta of all n particles:

$$\vec{p} = \vec{p}_1 + \vec{p}_2 + \cdots + \vec{p}_n = m_1 \vec{v}_1 + m_2 \vec{v}_2 + \cdots + m_n \vec{v}_n$$

- 2 From the motion of the CM, $\sum m_i \vec{v}_i = M \vec{v}_{CM}$. So:

$$\vec{p} = M \vec{v}_{CM}$$

TOTAL LINEAR MOMENTUM OF A SYSTEM

$$\vec{p} = M \vec{v}_{CM}$$

Total momentum of a system = **total mass** M $\times$ **velocity of the CM**. It is the same as the vector sum $m_1 \vec{v}_1 + m_2 \vec{v}_2 + \dots + m_n \vec{v}_n$.

• NEWTON'S SECOND LAW FOR A SYSTEM

DERIVATION

- 1 Differentiate $\vec{p} = M \vec{v}_{CM}$ with respect to time. The mass M is constant:

$$\frac{d\vec{p}}{dt} = M \frac{d}{dt} \vec{v}_{CM} = M \vec{a}_{CM}$$

- 2 From the motion of the CM, $M \vec{a}_{CM} = \vec{F}_{ext}$, the net external force. So:

$$\vec{F}_{ext} = \frac{d\vec{p}}{dt}$$

SECOND LAW FOR A SYSTEM

$$\vec{F}_{ext} = \frac{d\vec{p}}{dt}$$

The **total momentum** $\vec{p}$ of a system changes at a rate equal to the net **external** force $\vec{F}_{ext}$ on it.

• CONSERVATION OF LINEAR MOMENTUM

LAW

The **law of conservation of linear momentum** states that the total linear momentum of a system stays constant when no external force acts on it.

If $\vec{F}_{ext} = 0$, then $\frac{d\vec{p}}{dt} = 0$. So the total momentum does not change with time.

Momentum Conserved

$$\vec{p}_{initial} = \vec{p}_{final}$$

CM Velocity Constant

$\vec{F}_{ext} = 0$ and M is constant. So $M \vec{v}_{CM}$ is constant. The CM moves with **constant velocity**: $\vec{v}_{CM} = \text{constant}$.

Three Scalar Equations

$$p_x = C_1, p_y = C_2, p_z = C_3$$

p_x, p_y, p_z are the components of $\vec{p}$ along the x, y, z axes. C_1, C_2, C_3 are constants.

02

PART 2

Torque and Angular Momentum

2.1 Torque (Moment of Force)

DEFINITION

Torque is the **turning effect of a force** about a point or an axis. It is also called the **moment of force**.

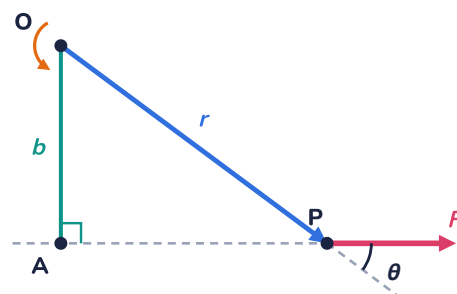
A force can push a body along its own direction. It can also **turn** the body about a point. Torque measures this turning effect.

The torque about a point O is the **vector product** of two vectors. The first is the position vector $\vec{r}$ of the point where the force acts, measured from O . The second is the force $\vec{F}$.

TORQUE (VECTOR FORM)

$$\vec{\tau} = \vec{r} \times \vec{F}$$

where, $\vec{r}$ = position vector, from O , of the point where $\vec{F}$ acts



Force $\vec{F}$ acts at P . The moment arm b is the perpendicular distance OA

MAGNITUDE OF TORQUE

$$|\vec{\tau}| = rF \sin \theta = F r_{\perp} = r F_{\perp}$$

where, θ = angle between $\vec{r}$ and $\vec{F}$

i Moment Arm $r_{\perp}$

$r_{\perp} = b = r \sin \theta$. It is the length of the **perpendicular** drawn from O to the line of action of $\vec{F}$. It is called the **moment arm**.

ii Perpendicular Force $F_{\perp}$

$F_{\perp} = F \sin \theta$ is the **component of $\vec{F}$ perpendicular to $\vec{r}$** . Only this part of the force turns the body.

• WHEN IS TORQUE ZERO?

$r = 0$: the force acts at O itself

$$\tau = 0$$

$F = 0$: no force acts

$$\tau = 0$$

$\theta = 0^\circ$ or 180° : $\vec{F}$ is parallel to $\vec{r}$

$$\tau = 0$$

• DIRECTION OF TORQUE

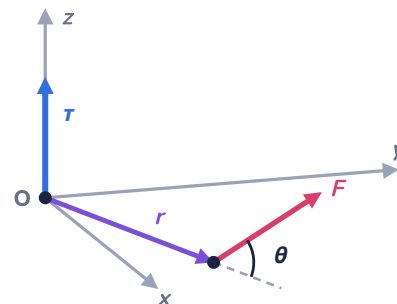
Torque is a vector product. So $\vec{\tau}$ is **perpendicular to the plane** that contains $\vec{r}$ and $\vec{F}$. The **right-hand rule** gives its direction.

Body turns **anticlockwise**

τ positive

Body turns **clockwise**

τ negative



$\vec{\tau}$ is perpendicular to the plane of $\vec{r}$ and $\vec{F}$

REMEMBER

Unit and Dimensions

The SI unit of torque is the **newton metre (N m)**. Its dimensions are $[ML^2T^{-2}]$. Work also has these **same dimensions**. Still, torque and work are **not the same quantity**.

2.2 Angular Momentum of a Particle

DEFINITION

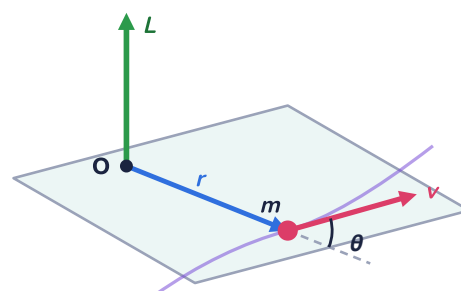
The **angular momentum** $\vec{L}$ of a particle about a point is the **moment of its linear momentum** about that point.

A particle of mass m moves with velocity $\vec{v}$. Its linear momentum is $\vec{p} = m\vec{v}$. Its angular momentum about O is the **vector product** of its position vector and its linear momentum.

ANGULAR MOMENTUM (VECTOR FORM)

$$\vec{L} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$$

where, $\vec{r}$ = position vector of the particle about O



$\vec{L}$ is perpendicular to the plane of $\vec{r}$ and $\vec{v}$

MAGNITUDE OF ANGULAR MOMENTUM

$$L = mvr \sin \theta = rp \sin \theta = r p_{\perp} = p r_{\perp}$$

where, θ = angle between $\vec{r}$ and $\vec{v}$ (or $\vec{p}$)

i Perpendicular Momentum $p_{\perp}$

$p_{\perp} = p \sin \theta$ is the **component of $\vec{p}$ perpendicular to $\vec{r}$** .

ii Perpendicular Distance $r_{\perp}$

$r_{\perp}$ is the length of the **perpendicular** from the origin O to the line along which $\vec{p}$ acts.

iii Direction

$\vec{L}$ is a vector product. So it is **perpendicular to the plane** of $\vec{r}$ and $\vec{p}$. The **right-hand rule** gives its direction, as for torque.

iv SI Unit

The SI unit of angular momentum is **J s**. This is the same as **kg m² s⁻¹**.

2.3 Relation between Torque and Angular Momentum

Force equals how fast linear momentum changes. In rotation, **torque equals how fast angular momentum changes**. This result is the **angular form of Newton's second law**.

• FOR A SINGLE PARTICLE

DERIVATION

1 Start with $\vec{L} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$. Differentiate it with respect to time, using the product rule:

$$\frac{d\vec{L}}{dt} = m \left(\frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt} \right)$$

2 Here $\frac{d\vec{r}}{dt} = \vec{v}$, and $\vec{v} \times \vec{v} = 0$. So the first term is zero. Also, $\frac{d\vec{v}}{dt} = \vec{a}$:

$$\frac{d\vec{L}}{dt} = m(\vec{r} \times \vec{a}) = \vec{r} \times (m\vec{a}) = \vec{r} \times \vec{F}$$

3 But $\vec{r} \times \vec{F}$ is the torque $\vec{\tau}$. So $\frac{d\vec{L}}{dt}$ equals the torque on the particle.

TORQUE-ANGULAR MOMENTUM (PARTICLE)

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

The angular momentum of a particle changes at a rate equal to the **torque** on it.

2.4 Torque and Angular Momentum of a System

Add the angular momenta of all the particles as **vectors**. This gives the total $\vec{L}$:

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \cdots + \vec{L}_n = \sum_{i=1}^n \vec{L}_i$$

DERIVATION

- 1 Differentiate with respect to time. For each particle, $\frac{d\vec{L}_i}{dt} = \vec{\tau}_i$. So the total gives the **net torque**:

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^n \frac{d\vec{L}_i}{dt} = \vec{\tau}_{net}$$

- 2 The force on each particle has two parts, external and internal. So the total torque also has two parts:

$$\vec{\tau} = \sum \vec{r}_i \times \vec{F}_i = \vec{\tau}_{ext} + \vec{\tau}_{int}$$

- 3 **Newton's third law** makes the internal torques cancel pair by pair. Hence $\vec{\tau}_{int} = 0$, and only the external torque is left.

TORQUE-ANGULAR MOMENTUM (SYSTEM)

$$\vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$$

Here $\vec{L}$ = total angular momentum of the system about a point, and $\vec{\tau}_{ext}$ = total **external torque** about that same point.

NOTE

Internal forces come in pairs of equal and opposite forces. The torques of each pair cancel. So the total angular momentum of a system can change only through **external torques**.

2.5 Conservation of Angular Momentum

LAW

The **law of conservation of angular momentum** states that the total angular momentum of a system stays constant when the net external torque on it is zero.

We know that $\vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$. If $\vec{\tau}_{ext} = 0$, then $\frac{d\vec{L}}{dt} = 0$. So $\vec{L}$ does not change with time.

CONSERVATION OF ANGULAR MOMENTUM

$$\vec{\tau}_{ext} = 0 \Rightarrow \vec{L} = \text{constant} \Rightarrow \vec{L}_{initial} = \vec{L}_{final}$$

For rotation about a fixed axis, $L = I\omega$, where I is the moment of inertia (see 4.1). So the working form is $I_1\omega_1 = I_2\omega_2$.

i All Components Conserved

If $\vec{L}$ is constant, each of its components is also constant:

$$L_x = \text{const}, L_y = \text{const}, L_z = \text{const}$$

ii Spin and Moment of Inertia

$L = I\omega$ stays constant. So when I **decreases**, ω **increases**. When I increases, ω decreases.

• APPLICATIONS OF THE LAW

Ice Skater / Ballet Dancer

A skater, a dancer or an acrobat in a circus spins faster by pulling the arms and legs **close to the body**. This **reduces** I , so ω **increases**. Stretching the arms out does the opposite.

Diver's Somersault

A diver leaves the board with the body stretched out. In the air, the diver **curls up** to reduce I . Since L is conserved, ω **increases** for the somersaults. Near the water, the diver stretches out again.

Helicopter's Two Propellers

A helicopter has **two propellers**. With only one, its body would **turn the opposite way** to conserve angular momentum. The second propeller stops this.

03

PART 3

Equilibrium and Centre of Gravity

3.1 Equilibrium of a Rigid Body

DEFINITION

Mechanical Equilibrium

A rigid body is in **mechanical equilibrium** when its **linear momentum and angular momentum both do not change** with time. Such a body has **no linear acceleration** and **no angular acceleration**.

For this, **two conditions** must hold at the same time:

(I) TRANSLATIONAL EQUILIBRIUM

$$\sum_{i=1}^n \vec{F}_i = 0$$

Net external force = 0. So $\sum F_{ix} = 0$, $\sum F_{iy} = 0$, $\sum F_{iz} = 0$. The linear momentum stays constant.

(II) ROTATIONAL EQUILIBRIUM

$$\sum_{i=1}^n \vec{\tau}_i = 0$$

Net external torque = 0. So $\sum \tau_{ix} = 0$, $\sum \tau_{iy} = 0$, $\sum \tau_{iz} = 0$. The angular momentum stays constant.

NOTE

- ◆ When the **net force is zero**, the condition of rotational equilibrium **does not depend on the choice of origin**.
- ◆ A body can also be in **partial equilibrium**. Then only one of the two conditions holds.

3.2 Partial Equilibrium

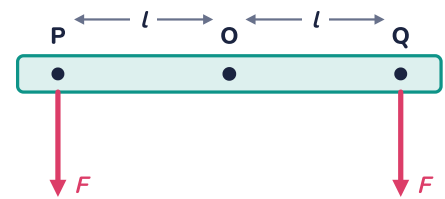
DEFINITION

Partial equilibrium is the state of a body in which **only one** of the two conditions of equilibrium holds.

Rotational Equilibrium Only

Here $\sum \vec{\tau} = 0$, but $\sum \vec{F} \neq 0$. So the body has **no angular acceleration**, but it does have a **linear acceleration**.

e.g. Two equal forces F act in the **same direction** at the ends P and Q of a rod. Both ends are at distance l from the centre O .



Net torque about O is zero, but the net force is not zero

Translational Equilibrium Only

Here $\sum \vec{F} = 0$, but $\sum \vec{\tau} \neq 0$. So the body has **no linear acceleration**, but it does have an **angular acceleration**.

e.g. Two equal and **opposite** forces act at the ends of the rod. They form a **couple**.



Net force is zero, but the couple gives a net torque

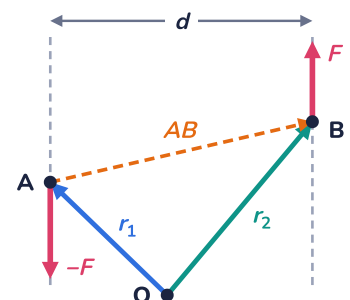
3.3 Couple

DEFINITION

A **couple** is a pair of **equal and opposite forces** whose lines of action are **parallel but not the same**. A couple produces **rotation without translation**.

Force $-\vec{F}$ acts at point A , whose position vector is $\vec{r}_1$. Force $+\vec{F}$ acts at point B , whose position vector is $\vec{r}_2$.

The total torque of the two forces is called the **moment of the couple**.



A couple: $-\vec{F}$ at A and $+\vec{F}$ at B

DERIVATION

- 1 Add the torques of the two forces about O :

$$\vec{\tau} = \vec{r}_1 \times (-\vec{F}) + \vec{r}_2 \times \vec{F} = (\vec{r}_2 - \vec{r}_1) \times \vec{F}$$

- 2 By the triangle law, $\vec{r}_1 + \vec{AB} = \vec{r}_2$. So $\vec{AB} = \vec{r}_2 - \vec{r}_1$. The moment depends only on $\vec{AB}$. So it **does not depend on the origin**.

MOMENT OF A COUPLE

$$\vec{\tau} = \vec{AB} \times \vec{F} \quad (\text{magnitude } \tau = Fd)$$

where, d = perpendicular gap between the lines of the two forces. A couple has the **same moment about every point**.

3.4 Principle of Moments

PRINCIPLE

The **principle of moments** states that the **algebraic sum of the torques** on a body in **rotational equilibrium** is zero.

Take **anticlockwise torques as positive** and **clockwise torques as negative**.

A light rod resting on a pivot is a **lever**. The pivot is called the **fulcrum**. A load F_1 acts at distance d_1 from the fulcrum. An effort F_2 acts at distance d_2 on the other side.

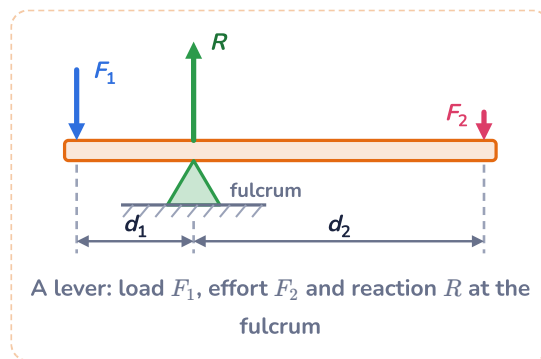
The lever is in equilibrium. So both conditions hold:

Translational equilibrium

$$R = F_1 + F_2$$

Rotational equilibrium (about the fulcrum)

$$F_1 d_1 = F_2 d_2$$



PRINCIPLE OF MOMENTS FOR A LEVER

$$F_1 d_1 = F_2 d_2 \quad (\text{load} \times \text{load arm} = \text{effort} \times \text{effort arm})$$

where, F_1 = load, d_1 = load arm, F_2 = effort, d_2 = effort arm

MECHANICAL ADVANTAGE

$$MA = \frac{F_1}{F_2} = \frac{d_2}{d_1}$$

MA > 1

We usually want a **mechanical advantage greater than one**. Then a **large load can be lifted with a small effort**.

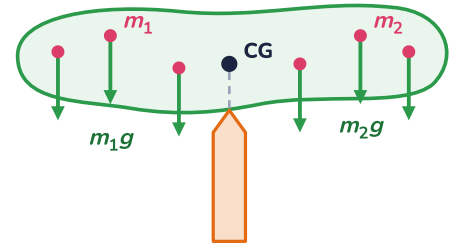
3.5 Centre of Gravity

DEFINITION

The **centre of gravity** (CG) of a body is the point about which the **total gravitational torque** on the body is **zero**. A body supported at its CG stays **balanced**.

Think of the body as many particles. A particle of mass m_i is at position vector $\vec{r}_i$ from the CG. Its weight is $m_i\vec{g}$.

The body balances on a sharp support placed at its CG. So the total gravitational torque about this point is zero.



A body balanced on a point support at its CG

DERIVATION

- 1 Total gravitational torque about the CG:

$$\vec{\tau}_g = \sum \vec{r}_i \times m_i \vec{g} = (\sum m_i \vec{r}_i) \times \vec{g} = 0$$

- 2 Suppose $\vec{g}$ is the **same at every point** of the body. Then $\sum m_i \vec{r}_i = 0$ about the CG. This is exactly the condition for the **centre of mass**. So the CG and the CM are the same point.

CG AND CM

$$\sum m_i \vec{r}_i = 0 \Rightarrow \text{CG coincides with CM}$$

This holds when g is uniform over the body.

i Small Bodies

g has the **same value** at every point of a small body. So its **CG and CM are at the same point**.

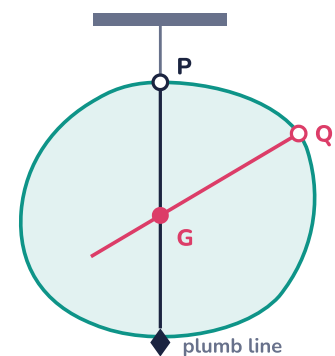
ii Very Large Bodies

g is **different** at different parts of a very large body. So its **CG and CM are at different points**.

PLUMB-LINE METHOD

CG of an Irregular Body

- 1 Hang the body freely from a point P . Mark the **vertical** through P with a plumb line.
- 2 Hang it from another point Q . Mark the new vertical.
- 3 The two verticals **meet at the CG**.



The two verticals meet at the CG, G

4.1 Moment of Inertia

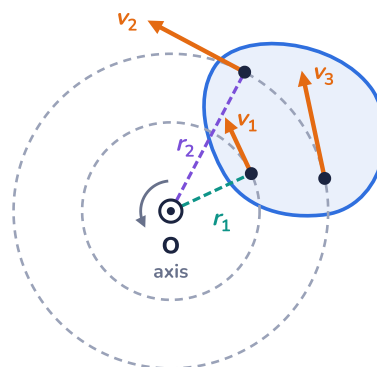
DEFINITION

The **moment of inertia** (I) of a body about an axis is its property to **resist any change** in its rotational state. This state may be rest or uniform rotation about that axis.

It is also called **rotational inertia**. In rotation, it plays the **same role as mass** plays in translation.

A rotating body has kinetic energy. Add the kinetic energies of all its particles to get it.

All the particles turn through the same angle in the same time. So they have the **same angular velocity** ω . A particle at distance r from the axis moves with speed $v = r\omega$.



Each particle moves on a circle about the axis
O with speed $v = r\omega$

DERIVATION

- Put $v_i = r_i\omega$ in the total kinetic energy of the particles:

$$K = \sum \frac{1}{2} m_i v_i^2 = \frac{1}{2} (\sum m_i r_i^2) \omega^2$$

- The quantity $\sum m_i r_i^2$ depends on how the mass is **spread about the axis**. This quantity is the moment of inertia I . So $K = \frac{1}{2} I \omega^2$.
- Compare this with the translational KE, $\frac{1}{2} m v^2$. So in rotation, I does the job that **mass** does in translation.

MOMENT OF INERTIA

$$I = \sum_{i=1}^n m_i r_i^2$$

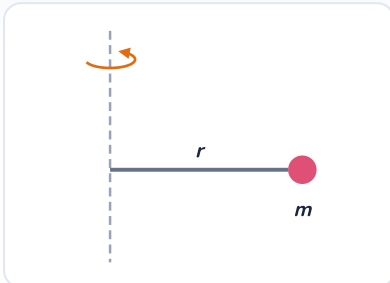
ROTATIONAL KINETIC ENERGY

$$K = \frac{1}{2} I \omega^2$$

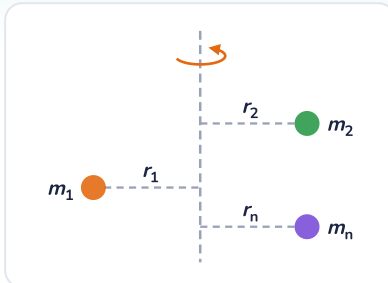
DEFINITION

For a rigid body turning about a fixed axis, the **moment of inertia** is the sum of mr^2 over all its particles. Here m is the mass of a particle and r is its distance from the axis.

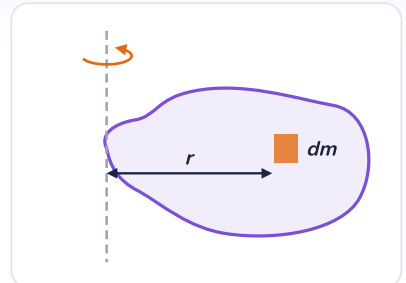
Moment of inertia for different kinds of mass distribution:

Single Particle

$$I = mr^2$$

System of Particles

$$I = \sum m_i r_i^2$$

Continuous Body

$$I = \int r^2 dm$$

NOTE**Rotational KE and Moment of Inertia**

Rotational KE is $K = \frac{1}{2} I \omega^2$. Put $\omega = 1$. Then $K = \frac{1}{2} I$, which means $I = 2 \times (\text{rotational KE})$.

So, for a body spinning with **unit angular velocity**, I has the **same numerical value as twice its rotational KE**.

• APPLICATIONS**Flywheel**

An engine shaft carries a heavy **flywheel**. Its **moment of inertia is large**. So it resists any sudden change in speed. The speed changes only **gradually**, and the ride is smooth, without jerks.

Bicycle / Cart Wheel

Most of the mass of the wheel is at its **rim**. Spokes join the rim to the axle. This **increases I** . So the wheel keeps turning for some time, even when the rider stops pedalling.

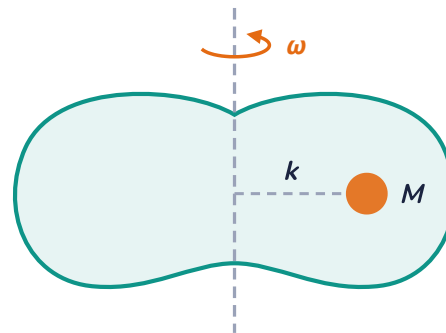
4.2 Radius of Gyration**DEFINITION**

The **radius of gyration** (k) of a body is that distance from the axis where its **whole mass**, if placed at one point, gives the **same moment of inertia** as the real body.

MOMENT OF INERTIA AND RADIUS OF GYRATION

$$I = Mk^2 \Rightarrow k = \sqrt{\frac{I}{M}}$$

where, M = total mass of the body



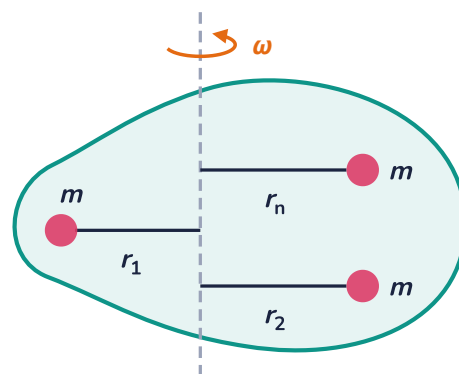
The whole mass M placed at distance k from the axis

● RADIUS OF GYRATION AS RMS DISTANCE

A body has n particles, each of mass m . Their distances from the axis are $r_1, r_2, \dots, r_n$. The total mass is $M = mn$.

NOTE

The value of k changes if the **axis moves**. It also depends on how the **mass is spread** around the axis. The total mass alone does not decide it.



Particles of a body at distances $r_1, r_2, \dots, r_n$ from the axis

DERIVATION

- 1 The moment of inertia of the particles is:

$$I = m(r_1^2 + r_2^2 + \dots + r_n^2)$$

- 2 By definition, $I = Mk^2$. Equate the two and use $M = mn$:

$$Mk^2 = m(r_1^2 + \dots + r_n^2) \Rightarrow k^2 = \frac{r_1^2 + \dots + r_n^2}{n}$$

RADIUS OF GYRATION (RMS DISTANCE)

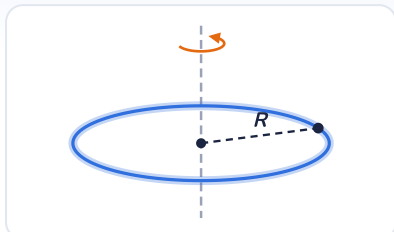
$$k = \sqrt{\frac{r_1^2 + r_2^2 + \dots + r_n^2}{n}}$$

So k is the **root-mean-square (RMS)** value of the distances of the particles from the axis.

4.3 Moment of Inertia of Standard Bodies

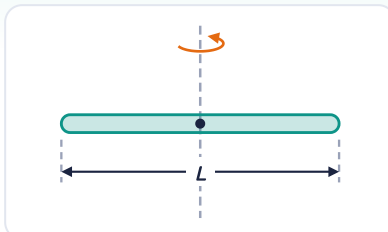
The cards give the **moment of inertia** I , the **radius of gyration** k and the ratio k^2/R^2 for some common bodies. Each value is about the axis shown by the **dashed line**.

Thin Circular Ring



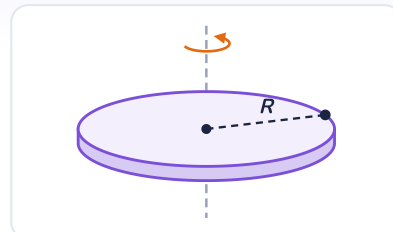
I	MR^2
k	R
k^2/R^2	1

Thin Rod



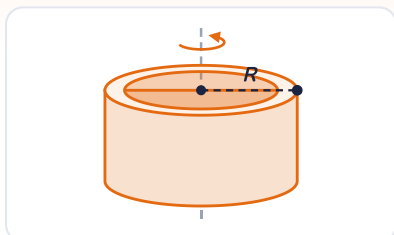
I	$\frac{1}{12}ML^2$
k	$\frac{L}{\sqrt{12}}$
k^2/R^2	—

Circular Disc



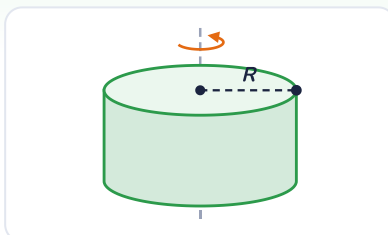
I	$\frac{1}{2}MR^2$
k	$\frac{R}{\sqrt{2}}$
k^2/R^2	$\frac{1}{2}$

Hollow Cylinder



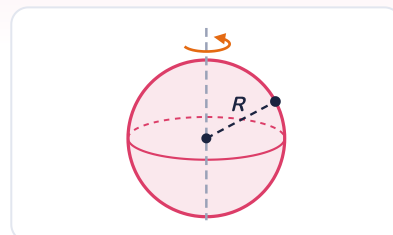
I	MR^2
k	R
k^2/R^2	1

Solid Cylinder



I	$\frac{MR^2}{2}$
k	$\frac{R}{\sqrt{2}}$
k^2/R^2	$\frac{1}{2}$

Solid Sphere



I	$\frac{2}{5}MR^2$
k	$\sqrt{\frac{2}{5}}R$
k^2/R^2	$\frac{2}{5}$

NOTE

- ◆ The moment of inertia of a body **depends on the axis** chosen. All the values above are for the axes shown.
- ◆ For the ring, the disc, the two cylinders and the sphere, the axis passes **through the centre**. For the rod, it passes through the centre, **perpendicular to its length**.
- ◆ The **ring and the hollow cylinder** have the largest ratio, $k^2/R^2 = 1$. All their mass is at distance R from the axis.
- ◆ The **solid sphere** has the smallest ratio, $\frac{2}{5}$.

4.4 Kinematics of Rotational Motion

Let a body turn about a **fixed axis**, and let its **angular acceleration stay constant**. Its equations of motion have the **same form as the equations of linear motion**. The rotational quantities simply take the place of the linear ones.

Linear Motion

$$v = u + at$$

$$s = ut + \frac{1}{2}at^2$$

$$v^2 - u^2 = 2as$$

Rotational Motion

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega^2 - \omega_0^2 = 2\alpha(\theta - \theta_0)$$

Here θ_0 and ω_0 are the **initial angular position** and the **initial angular velocity** of the body. If $\theta_0 = 0$, the second equation becomes $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$.

The linear and rotational quantities match like this:

QUANTITY	LINEAR	ROTATIONAL
Displacement	s	$\theta - \theta_0$ (angular displacement)
Initial velocity	u	ω_0 (initial angular velocity)
Final velocity	v	ω (angular velocity)
Acceleration	a	α (angular acceleration)
Time	t	t

NOTE

Use these equations only if α is **constant**. The linear equations, too, need a constant acceleration.

4.5 Dynamics of Rotational Motion

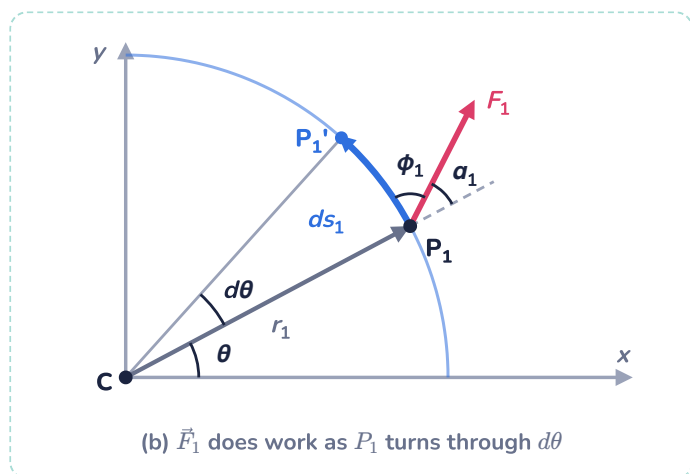
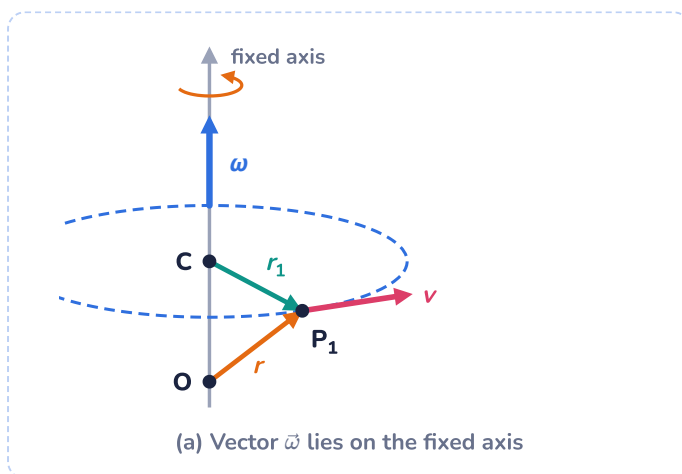
Each quantity in translation has a partner quantity in rotation. This partner is its **rotational analogue**. The table compares them for motion about a fixed axis.

QUANTITY	PURE TRANSLATION	PURE ROTATION
Position	x	Angular position, θ
Velocity	$v = \frac{dx}{dt}$	$\omega = \frac{d\theta}{dt}$
Acceleration	$a = \frac{dv}{dt}$	$\alpha = \frac{d\omega}{dt}$
Inertia	Mass, m	Moment of inertia, I
Newton's second law	$F = ma$	$\tau = I\alpha$

QUANTITY	PURE TRANSLATION	PURE ROTATION
Work done	$W = \int F dx$	$W = \int \tau d\theta$
Kinetic energy	$K = \frac{1}{2}mv^2$	$K = \frac{1}{2}I\omega^2$
Power	$P = Fv$	$P = \tau\omega$
Momentum	$p = mv$	$L = I\omega$

4.6 Work Done by a Torque and Power

Let a rigid body turn about a **fixed axis**. A force $\vec{F}_1$ acts on one of its particles at point P_1 . Its line of action is in a plane that is **at right angles to the axis**.



DERIVATION

- P_1 moves on a circle of radius $r_1 = CP_1$ with centre C on the axis. When the body turns through a small angle $d\theta$, P_1 moves through $ds_1 = r_1 d\theta$. So the work done by $\vec{F}_1$ is:

$$dW_1 = \vec{F}_1 \cdot d\vec{s}_1 = F_1 ds_1 \cos \phi_1 = F_1 (r_1 d\theta) \sin \alpha_1$$

ϕ_1 = angle between $\vec{F}_1$ and the tangent at P_1 . α_1 = angle between $\vec{F}_1$ and the radius CP_1 . So $\phi_1 + \alpha_1 = 90^\circ$.

- The torque of $\vec{F}_1$ about the axis is $\vec{\tau}_1 = \vec{CP}_1 \times \vec{F}_1$. Its magnitude is $\tau_1 = r_1 F_1 \sin \alpha_1$. The part OC of $\vec{OP}_1$ lies along the axis, so it gives no torque about the axis. Therefore:

$$dW_1 = \tau_1 d\theta$$

- Now add the work done by all the forces. Their torques add up to the net torque $\tau = \tau_1 + \tau_2 + \dots$ on the body. So $dW = (\tau_1 + \tau_2 + \dots) d\theta = \tau d\theta$. Divide both sides by dt :

$$\frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau\omega \quad (\because \omega = \frac{d\theta}{dt})$$

WORK DONE BY A TORQUE

$$W = \int \tau d\theta = \tau \theta$$

The last form is for a constant torque.

INSTANTANEOUS POWER

$$P = \frac{dW}{dt} = \tau \omega$$

NOTE

For rotation about a **fixed axis**, take only the **forces that lie in planes perpendicular to the axis**. Also take only the **parts of the position vectors perpendicular to the axis**. Ignore the parts of torque that are perpendicular to the fixed axis.

4.7 Relation between Torque and Moment of Inertia

A rigid body has rotational KE, $K = \frac{1}{2}I\omega^2$. If the body has an **angular acceleration** α , this KE keeps changing. The power is the rate of this change.

DERIVATION

- 1 Power = rate of change of rotational KE. Here I is constant and $\alpha = \frac{d\omega}{dt}$:

$$P = \frac{d}{dt} \left(\frac{1}{2}I\omega^2 \right) = \frac{1}{2}I(2\omega) \frac{d\omega}{dt} = I\omega\alpha$$

- 2 The power of a torque is also $P = \tau\omega$ (see 4.6). Equate the two:

$$\tau\omega = I\omega\alpha \Rightarrow \tau = I\alpha$$

NEWTON'S SECOND LAW FOR ROTATION

$$\tau = I\alpha$$

Net torque = moment of inertia $\times$ angular acceleration. It is like Newton's second law, $F = ma$, for translation with constant mass.

4.8 Angular Momentum about a Fixed Axis

About a **fixed axis**, the angular momentum of a rigid body has a simple form, like $p = mv$.

DERIVATION

- 1 A particle of mass m_i at distance r_i from the axis has speed $v_i = r_i\omega$. Its angular momentum is $m_iv_ir_i$. So the total is:

$$L = \sum m_iv_ir_i$$

2 Put $v_i = r_i \omega$. All the particles have the same ω :

$$L = \sum m_i (r_i \omega) r_i = (\sum m_i r_i^2) \omega$$

3 But $\sum m_i r_i^2 = I$, the moment of inertia about the axis. So $L = I\omega$.

ANGULAR MOMENTUM ABOUT A FIXED AXIS

$$L = I\omega$$

This is the rotational analogue of linear momentum, $p = mv$.

i Relation to Torque

Differentiate, keeping I constant:

$$\frac{dL}{dt} = \frac{d(I\omega)}{dt} = I\alpha = \tau$$

ii Conservation

If $\tau_{ext} = 0$, then $L = I\omega$ stays constant:

$$I_1 \omega_1 = I_2 \omega_2$$

NOTE

For one particle, $\vec{L}$ and $\vec{\omega}$ **need not point the same way**. Linear momentum and velocity, on the other hand, always point the same way. If a **symmetric body** turns about a fixed axis, $\vec{L}$ lies along the axis and $\vec{L} = I\vec{\omega}$.

4.9 Rolling Motion

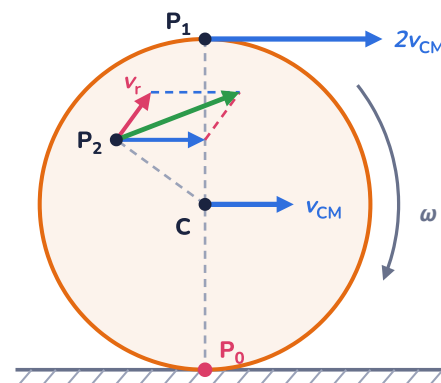
DEFINITION

Rolling motion is a mix of **pure translation and pure rotation** happening together. It is very common in daily life. For example, the wheels of a vehicle roll on the road.

In rolling **without slipping**, the point of the body that touches the surface has **zero velocity** at every instant.

A disc rolls on a level surface. Its centre of mass C is its geometric centre. C moves with velocity v_{CM} parallel to the surface.

At the same time, the disc spins about the axis through C . So each point has two velocities at once: v_{CM} and the rotational velocity v_r . Its actual velocity is their **vector sum**. Here $v_r = r\omega$, and v_r is perpendicular to r .



A disc rolling to the right: P_0 is at rest and the top point P_1 moves at $2v_{CM}$

DERIVATION

- 1 At the contact point P_0 , the rotational velocity $R\omega$ points **opposite** to v_{CM} .
- 2 There is no slipping, so P_0 is **at rest**. The two velocities must cancel:

$$v_{CM} - R\omega = 0 \Rightarrow v_{CM} = R\omega$$

CONDITION FOR ROLLING WITHOUT SLIPPING

$$v_{CM} = R\omega$$

where, R = radius of the rolling body, ω = angular velocity of the body

Velocity of the **contact point** P_0

0 (at rest)

Velocity of the **centre** C

$$v_{CM} = R\omega$$

Velocity of the **topmost point** P_1

$$v_{CM} + R\omega = 2v_{CM}$$

• KINETIC ENERGY OF A ROLLING BODY

DERIVATION

- 1 A body rolling without slipping has **translational KE** and **rotational KE**. Its total KE is their sum:

$$K = \frac{1}{2}mv_{CM}^2 + \frac{1}{2}I\omega^2$$

- 2 Put $I = mk^2$ and $\omega = \frac{v_{CM}}{R}$. Here k is the radius of gyration:

$$K = \frac{1}{2}mv_{CM}^2 + \frac{1}{2}mk^2 \left(\frac{v_{CM}}{R} \right)^2$$

- 3 Take $\frac{1}{2}mv_{CM}^2$ common to get the final result.

KINETIC ENERGY OF A ROLLING BODY

$$K = \frac{1}{2}mv_{CM}^2 \left(1 + \frac{k^2}{R^2} \right)$$

where, $\frac{k^2}{R^2}$ is the ratio given for each body in topic 4.3